Comments on LaPlace

Fourier transform does not apply to all classes of signals that we need to work with. Generalization from Fourier to LaPlace is accomplished by multiplying a signal $x(t)$ by an exponential convergence factor and Fourier transforming the product.

Consider only signals that are zero for $t<0$. Then an appropriate factor is $e^{-\sigma t}$ where σ is positive.

This gives the Fourier Transform

$$X(\sigma + j\omega) = \int_0^\infty x(t) e^{-\sigma t} e^{-j\omega t}\, dt$$

$$= \int_0^\infty x(t)\, e^{-(\sigma + j\omega)t}\, dt$$

let $s = \sigma + j\omega$ then

$$X(s) = \int_0^\infty x(t) e^{-st}\, dt \quad \text{denoted by } \mathcal{L}[x(t)]$$

where $\mathcal{L}[\bullet]$ denotes the operation of the single-sided Laplace transform.

For example let $x(t) = 1$

$$X(s) = \int_0^\infty e^{-st}\, dt = \frac{1}{-s} e^{-st}\Big|_0^\infty = \frac{e^{-\sigma t} e^{-j\omega t}}{-s}\Big|_0^\infty$$

$$= \frac{-e^{-\sigma t}}{s}\left(\cos \omega t - j \sin \omega t\right)\Big|_0^\infty$$

if $Re(s) = \sigma > 0$ then $X(s) = \frac{1}{s}$

but if $\sigma < 0$ then does not converge

Transforms of Derivatives: the Laplace of $dx(t)/dt$ is

$$\mathcal{L}\left[\frac{dx(t)}{dt}\right] = \int_0^\infty \frac{dx(t)}{dt} e^{-st}\, dt$$

integration by parts gives $\quad u = e^{-st}$ and $dv = dx(t)$

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$

$du = -s\,e^{-st}$ and $v = \cancel{x}(t)$

$$\int_0^\infty \frac{dx(t)}{dt} e^{-st} dt = \left[e^{-st} x(t)\right]_0^\infty + s\int_0^\infty x(t)\, e^{-st}$$

$$= s\,X(s) - x(0)$$

if $\lim_{t \to \infty} x(\cancel{t})\, e^{-st} = 0$

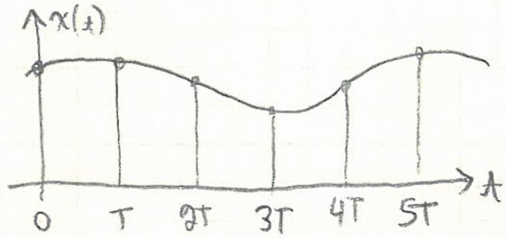
recall that for an inductor $v(t) = L\frac{di}{dt}$

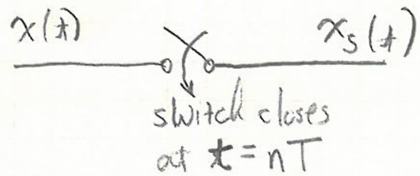
$$V(s) = sL\,I(s) - L\,i(0)$$

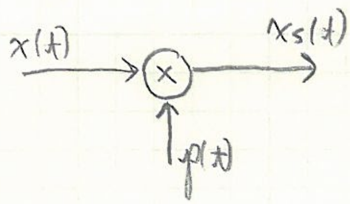
$$I(s) = \frac{1}{sL} V(s) + \frac{1}{s} i(0)$$

<u>Z-transform</u>

Suppose we sample a signal $x(t)$ at regular intervals T

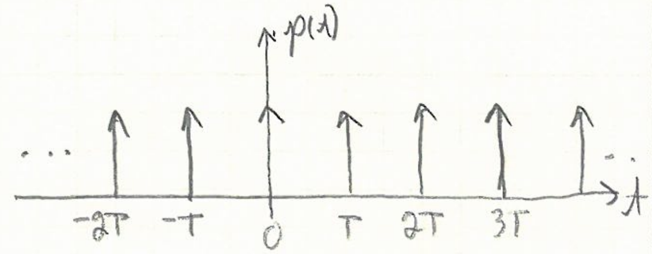






model of sampler

$$x_s(t) = x(t)\,p(t)$$



if switch close time is very small relative to variation of $x(t)$ we can assume that $p(t)$ is the δ

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\therefore \quad x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(t)\,\delta(t - kT)$$

Since $\delta(t - kT)$ is zero except at $t = kT$ then $x(t)$ can be replaced by $x(kT)$ if $x(t)$ is continuous at $t = kT$.

also assume $x(t) \equiv 0 \quad t < 0$

then $x_s(t) = \sum_{k=0}^{\infty} x(kT)\,\delta(t - kT)$

Taking Laplace yields $\quad X_s(s) = \int_0^{\infty} \sum_{k=0}^{\infty} x(kT)\,\delta(t - kT)\, e^{-st}\, dt$

$$X_s(s) = \sum_{k=0}^{\infty} x(kT) \int_0^{\infty} \delta(t-kT) e^{-st}\, dt$$

$$X_s(s) = \sum_{k=0}^{\infty} x(kT) e^{-skT}$$

now define a new complex variable z as $z = e^{sT}$

then $X_s(z) = \sum_{k=0}^{\infty} x(kT) z^{-k}$

$X_s(z)$ is known as the z-transform of sequence $x(kT)$

The coefficient $x(kT)$ denotes the sample value and z^{-k} denotes that sample occurs k sample periods after the $t=0$ reference. Note that e^{sT} is the T-seconds time shift operator. The sample value $x(kT)$ is multiplied by e^{-skT} which places that sample value at $t=kT$. So z is simply a shorthand notation for the Laplace time shift operator at discrete sample times.

Ex: $30.5z^{-20}$ denotes a sample with value 30.5 which occurs 20 sample periods after $t=0$.

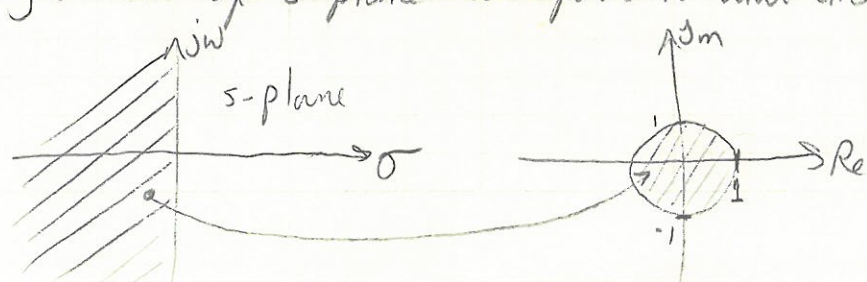
Recall that $s = \sigma + jw$ where σ is a constant to ensure convergence

$z = e^{\sigma T} e^{jwT}$

$|z| = e^{\sigma T}$ then $\sigma > 0$ (right half of s-plane) corresponds to $|z| > 1$ and

$\sigma < 0$ (left half of s-plane) maps to $|z| < 1$

jw axis of s-plane corresponds to unit circle of z-plane



Consider the unit step sample sequence

$$x(kT) = 1 \quad , \quad k \geq 0$$

$$X(z) = \sum_{k=0}^{\infty} z^{-k} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \ldots$$

recall that for the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x| < 1$$

since

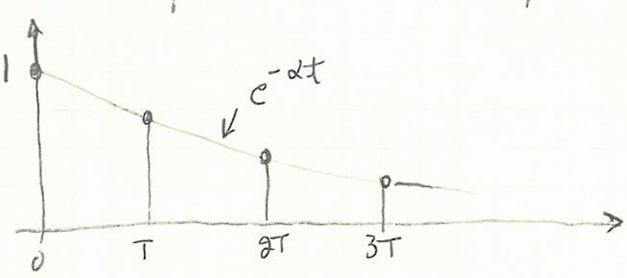
$$S_g = 1 + x + x^2 + x^3 + x^4 + \ldots$$

$$x S_g = x + x^2 + x^3 + x^4 + x^5 + \ldots$$

$$S_g - x S_g = 1 \quad \therefore \quad S_g = \frac{1}{1-x}$$

then $$X(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} \qquad |z| > 1$$

As another example consider the sampled exponential $e^{-\alpha t}$



$$x(kT) = e^{-\alpha kT} \qquad \alpha > 0, \; k \geq 0$$

$$X(z) = \sum_{k=0}^{\infty} e^{-\alpha kT} z^{-k} = \sum_{k=0}^{\infty} \left(e^{-\alpha T} z^{-1}\right)^k$$

then

$$X(z) = \frac{1}{1 - e^{-\alpha T} z^{-1}} \qquad |z| > e^{-\alpha T}$$

if α is fixed and for a constant sampling period

$$M = e^{-\alpha T}$$

$$X(z) = \frac{1}{1 - M z^{-1}} \qquad |z| > M$$

$$= \frac{z}{z - M}$$

$\rightarrow X(z)$ has a zero at $z = 0$ and pole at $z = M$

$\therefore$ only for $|M| < 1$ does M^k go to zero for large k.